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Symmetric linear Bäcklund transformation for discrete BKP and DKP equations

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Abstract. Proper lattices for the discrete BKP and DKP equations are determined. Linear Bäcklund transformation equations for the discrete BKP and DKP equations are constructed, which possess the lattice symmetries and generate auto-Bäcklund transformations.

1. Introduction

The discrete counterpart of the KP equation, known as the discrete KP equation or Hirota bilinear difference equation (HBDE) [4, 5, 8], plays a central role in the study of integrable nonlinear systems. It embodies an infinite number of integrable differential equations. It is satisfied by the string correlation function in particle physics [2], and is also satisfied by the transfer matrices of some solvable lattice model [9].

The equation possesses the auto-Bäcklund transformation which acts on the solutions of the discrete KP equation and turns the n-soliton solution into the $(n + 1)$-soliton solution [8]. This transformation is generated by a pair of linear equations which I call the linear Bäcklund transformation equations (LBTE) in this paper. They also play an important role in the study of integrable systems. They are the Lax pair of the discrete KP equation, and generate the Bethe ansatz solution of some solvable lattice model [2].

On the other hand, from consideration of the structure of the discrete KP equation, the dependent variables of the discrete KP equation are believed to reside on a face centred cubic (FCC) lattice. Indeed, the discrete KP equation does not change its form under rotation of the FCC lattice. However, the LBTE to the discrete KP equation changes those forms under that rotation.

In a previous paper [1], the LBTE are extended to possess the lattice symmetry, which I call the symmetric LBTE in this paper, and the following results are obtained. The extended equations were arranged in the form matrix $\times$ vector $= 0$ by considering the lattice symmetry. The condition that the extended equations have nontrivial solutions, i.e. vanishing of the determinant of the matrix, is just the discrete KP equation. Furthermore, the extended equations were generalized to higher dimensions due to the lattice symmetry itself.

On the one hand, comparing with the KP hierarchy which possesses A_∞-type Lie group symmetry acting on the space of solutions [5], there are integrable hierarchies which possess the B_∞ or D_∞ Lie group symmetries [3,6]. Such hierarchies are known as the BKP, fermionic BKP and DKP hierarchies. For these hierarchies, one can find discrete equations corresponding to the discrete KP equation by considering the infinite-dimensional symmetry behind them. What happens when one corresponds proper lattices to the discrete equations, and considers the lattice symmetry of the LBTE to these discrete equations?

In this paper, I show the following results obtained by considering the above question.

First, proper lattices are determined by comparing the type of the infinite-dimensional symmetries and the dimensions of the discrete equations. The discrete equations are invariant under rotation of the lattices.

Second, the LBTE to the discrete equation which possesses the lattice symmetry are constructed. The LBTE can be represented by the functions on the vertices of some regular polyhedrons in the lattice.

Third, only the discrete equation is derived from the consistency condition for the LBTE, when one considers the following two types of consistency conditions. One condition is the determinant-type consistency condition explained above, and the other is a condition similar to the compatibility condition for the Lax pair, which is called the compatibility-type condition in this paper. In some cases, the LBTE cannot be arranged in the form matrix $\times$ vector $= 0$, however, in such cases, the compatibility-type condition becomes just the discrete equation. Furthermore, by considering these two types of conditions, one can check that the LBTE generates the auto-Bäcklund transformation for the discrete equation.

After explaining the main ideas behind the method for the case of the KP hierarchy in section 2, I show the above results in the case of the BKP, fermionic BKP and DKP hierarchies in sections 3 and 4.

2. The KP hierarchy case

The A_∞ symmetry space of solutions to the KP hierarchy allows one to deal with this hierarchy in a simple form [7]. This section is devoted to summarizing some results obtained in our previous paper [1], concerning the discrete formula for the KP hierarchy and its LBTE.

The lowest-dimensional discrete formula belonging to the KP hierarchy, known as the discrete KP equation, is a three-dimensional equation [4, 5, 8] (also see the appendix). The equation is the most fundamental equation among the discrete formulae for the KP hierarchy, in the sense that higher-dimensional discrete formulae can be decomposed into it. The discrete KP equation is presented as follows:

$$\begin{aligned} z_{12}z_{34}&f(k_1+1,k_2+1,k_3,k_4)f(k_1,k_2,k_3+1,k_4+1)\\ &-z_{13}z_{24}f(k_1+1,k_2,k_3+1,k_4)f(k_1,k_2+1,k_3,k_4+1)\\ &+z_{14}z_{23}f(k_1+1,k_2,k_3,k_4+1)f(k_1,k_2+1,k_3+1,k_4)=0. \end{aligned} \tag{1}$$

Here, z_{ij} is

$$z_{ij}=z_i-z_j \tag{2}$$

and $z_i (i = 1, 2, 3, 4)$ are arbitrary complex constants. The equation seems to be defined on a four-dimensional lattice space. However, since the sum of the four variables does not vary in the equation, the equation is actually a three-dimensional equation. Indeed, the following variable transformation allows one to represent the equation using three variables:

$$p=k_1 \qquad q=k_2 \qquad r=k_3 \qquad n=k_1+k_2+k_3+k_4. \tag{3}$$

First, I seek the proper lattice for the discrete KP equation. To find it, I relate a regular polyhedron to the discrete KP equation, and join the polyhedrons. The proper polyhedron is considered to be a octahedron, because the equation connects f at six points. In fact, when one considers the four variables to be orthogonal, the equation takes the form of a summation of the products of f on one vertex of the octahedron and another f on the opposite side vertex. By joining octahedrons, a FCC lattice is constructed. Thus, the proper lattice for the discrete KP equation turns out to be a FCC lattice.

It is known that there exist equations which generate the Bäcklund transformation [2, 8]. In a previous paper [1], the equations are extended to possess the lattice symmetry of a FCC lattice. Each of the equations takes the same form as the discrete formula of the modified KP hierarchy [5] (see the appendix). The equations take the following form:

$$\sum_{j,k,l=1}^{4} \tfrac{1}{2}\epsilon_{ijkl} z_{kl} f_{kl} g_j = 0 \qquad i = 1, 2, 3, 4. \tag{4}$$

Where, ϵ_{ijkl} is the Levi–Civita tensor, and small indices on f and g increase the variable corresponding to the index by one.

For example,

$$\begin{aligned} &g_1 = g(k_1 + 1, k_2, k_3, k_4) \qquad f_{12} = f(k_1 + 1, k_2 + 1, k_3, k_4) \\ &f_{11} = f(k_1 + 2, k_2, k_3, k_4). \end{aligned} \tag{5}$$

In our previous paper I called these equations the symmetric LBTE.

The symmetric LBTE can be related to the fundamental regular polyhedrons in the FCC lattice, when one suitably arranges these equations. The fundamental regular polyhedrons in the FCC lattice are an octahedron and two tetrahedrons. There are two such arrangements, one of which is as follows:

$$\begin{pmatrix} 0 & z_{34}f_{34} & -z_{24}f_{24} & z_{23}f_{23} \\ -z_{34}f_{34} & 0 & z_{14}f_{14} & -z_{13}f_{13} \\ z_{24}f_{24} & -z_{14}f_{14} & 0 & z_{12}f_{12} \\ -z_{23}f_{23} & z_{13}f_{13} & -z_{12}f_{12} & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = 0. \tag{6}$$

Another arrangement is to construct the matrix and vector elements with g and f. In both of the arrangements, f and g in the matrix elements are on the vertices of an octahedron. On the other hand, each g and f in the vector elements are on the vertices of a tetrahedron dual to one another. Note that the arrangements make the lattice symmetry of the equations manifest.

One should also note the following. The FCC lattice is a root lattice of the A_3-type Lie group, and the two tetrahedrons and one octahedron are weight diagrams of three fundamental representations of the A_3-type Lie group. This implies that one can determine the regular polyhedrons and proper lattice for the discrete formula, by comparing the dimensions of the equation and symmetric groups corresponding to the hierarchy.

In what follows, I show that symmetric LBTE generate the Bäcklund transformation successfully. Here, 'successfully' means the following: whenever f satisfies the discrete KP equation, the symmetric LBTE can be solved for g, and the solution solves the discrete KP equation automatically.

First, both f and g in the symmetric LBTE satisfy the discrete KP equation when they satisfy the symmetric LBTE. To show this, we consider equation (6) as four linear equations to be solved for four g values. These four linear equations can only be solved if the determinant of the coefficient matrix vanishes. In this case, one can use the fact that determinant of the antisymmetric matrix is the square of a Pfaffian. This fact leads the condition to the following compact form:

$$\begin{aligned} \mathrm{Det}(a_{ij}) &= (\mathrm{Pfaff}(a_{ij}))^2 \\ &= (z_{12}z_{34}f_{12}f_{34} - z_{13}z_{24}f_{13}f_{24} + z_{14}z_{23}f_{14}f_{23})^2 = 0. \end{aligned} \tag{7}$$

This is simply the discrete KP equation. Now, one can exchange the roles of f and g, by translating each equation in (6), so that g also satisfies the discrete KP equation. This explanation depends essentially on the arrangement of the symmetric LBTE.

Second, the symmetric LBTE can be solved for g whenever f satisfies the discrete KP equation. This fact and the first statement show that symmetric LBTE generate the Bäcklund transformation successfully. However, it requires a little more consideration. In fact, consistency conditions other than the determinant-type consistency condition arise as follows. Consider that the determinant-type consistency condition is already satisfied. Then, since the rank of the coefficient matrix is 2, two out of four equations in (6) remain independent. I chose two such equations as the first two equations in (6). For simplicity, I express the equations using the variables p, q, r in (3):

$$\begin{aligned} 1: &\quad z_{34} f_r g_q - z_{24} f_q g_r + z_{23} f_{qr} g = 0 \\ 2: &\quad -z_{14} f_p g_r + z_{13} f_{pr} g + z_{34} f_r g_p = 0. \end{aligned} \tag{8}$$

These two equations provide several methods to obtain g on one point from g on other points. The value of g obtained needs to be independent of the method chosen. For example, in figure 1, one can obtain the value of $g(H)$ from $g(A)$, $g(B)$, $g(C)$, according to the following two procedures:

$$\begin{aligned} 1: &\quad g(A), g(B) \xrightarrow{1} g(F) \\ &\quad g(B), g(C) \xrightarrow{1} g(G) \qquad \xrightarrow{2} \qquad g(F), g(G) \rightarrow g(H) \\ 2: &\quad g(A), g(B) \xrightarrow{2} g(D) \\ &\quad g(B), g(C) \xrightarrow{2} g(E) \qquad \xrightarrow{1} \qquad g(D), g(E) \xrightarrow{1} g(H). \end{aligned} \tag{9}$$

One can evaluate the value of g on one point from the values of g on two points by using the equation corresponding to the number given over the arrow (8). Here, the arrow represents that process. After some calculation, the two procedures lead to the following expressions for $g(H)$:

$$\begin{aligned} 1: \quad g(H) &= g(A) \frac{z_{14} z_{24} f_{pq} f_{qr}}{z_{34} z_{34} f_{qr} f_{rr}} \\ &\quad + g(C) \frac{z_{31} z_{32} f_{pqr} f_{qr}}{z_{34} z_{34} f_{qr} f_{rr}} + g(B) \left(\frac{z_{14} z_{32} f_{pq} f_{qrr}}{z_{34} z_{34} f_{qr} f_{rr}} + \frac{z_{31} z_{24} f_{pqr} f_q}{z_{34} z_{34} f_{qr} f_r} \right) \\ 2: \quad g(H) &= g(A) \frac{z_{14} z_{24} f_{pq} f_{pr}}{z_{34} z_{34} f_{pr} f_{rr}} \\ &\quad + g(C) \frac{z_{31} z_{32} f_{pqr} f_{pr}}{z_{34} z_{34} f_{pr} f_{rr}} + g(B) \left(\frac{z_{24} z_{31} f_{pq} f_{prr}}{z_{34} z_{34} f_{pr} f_{rr}} + \frac{z_{32} z_{14} f_{pqr} f_p}{z_{34} z_{34} f_{pr} f_r} \right). \end{aligned} \tag{10}$$

These two expressions for $g(H)$ must coincide. By equating the coefficients of $g(B)$, one can express the condition as follows:

$$\frac{f_{pq} f_{pqr}}{z_{34}^2 f_{qr} f_{pr}} \times (\mathrm{e}^{\partial_r} - 1) \left(\frac{\text{Discrete } KP}{f_r f_{pq}} \right) = 0. \tag{11}$$

Here, e^{∂_i} acts on arbitrary functions from the left-hand side and increases the variable corresponding to the index i by one in the function. In the equation discrete KP expresses the right-hand side of equation (1). The coefficients of $g(A)$, $g(C)$ automatically coincide. Thus, in this case, the condition is satisfied if f satisfies the discrete KP equation.

However, this compatibility condition guarantees the existence of the solution of g to the symmetric LBTE. To show this, I first put the initial value of g on the plane $p + q = c$. Here c is arbitrary constant. Using the two equations in (10) independently, one can obtain the g

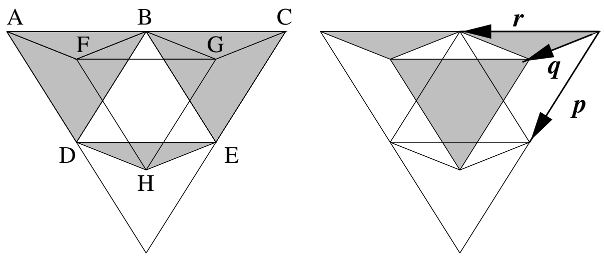

Figure 1. Compatibility condition for discrete KP equation.

values on the plane $p + q = c + 1$. To make the value of g on the plane $p + q = c + 1$ unique, I impose the following condition on the initial value of g:

$$\frac{-z_{24} f_{pq} g_{pr} + z_{23} f_{pqr} g_p}{z_{34} f_{pr}} = \frac{-z_{14} f_{pq} g_{qr} + z_{13} f_{pqr} g_q}{z_{34} f_{qr}}. \tag{12}$$

If the compatibility condition is satisfied, this condition is also satisfied by g on the plane $p + q = c + 1$. Therefore, one can inductively construct g on all lattice points, if the compatibility condition is satisfied. One can conclude that there exists at least one solution of g for the symmetric LBTE with f being a solution to the discrete KP equation.

3. The BKP and fermionic BKP hierarchies case

The lowest-dimension discrete formulae belonging to the BKP and the fermionic BKP hierarchies are known to be as follows [3, 4, 6] (see the appendix):

$$a_{12}b_{23}a_{31} f_1 f_{23} + a_{12}a_{23}b_{31} f_2 f_{13} + b_{12}a_{23}a_{31} f_3 f_{12} + b_{12}b_{23}b_{31} f f_{123} = 0. \tag{13}$$

Here, a_{ij} and b_{ij} are constant coefficients, and can be expressed as

$$\begin{array}{lll} \text{BKP} & a_{ij} = z_i + z_j & b_{ij} = z_i - z_j \\ \text{fermionic BKP} & a_{ij} = 1 & b_{ij} = z_i - z_j \end{array} \tag{14}$$

respectively. Here, $z_i (i = 1, 2, 3)$ are arbitrary complex constants. The equation is called the discrete BKP equation, for the first choice of coefficient in (14). However, I call the equation the discrete BKP equation for both cases, neglecting the difference between the coefficients.

By considering the modified BKP hierarchy and the modified fermionic BKP hierarchy [3, 6], one can obtain the LBTE for the equations. Collecting the discrete formulae which depend on the variables appearing in the discrete BKP equation, in the modified BKP hierarchy, the following three equations can be obtained (see the appendix):

$$a_{ij}(f_j g_i - f_i g_j) = b_{ij}(f g_{ij} - f_{ij} g) \qquad i, j = 1, 2, 3 \quad i \neq j. \tag{15}$$

The discrete BKP equation is a three-dimensional equation and connects f at eight points. Thus the cube may become a proper regular polyhedron for the discrete BKP equation. Indeed, when the three variables p, q, r are considered to be orthogonal, the equation takes the form of the summation of products of f on one vertex of the cube and another f on the opposite side vertex. By joining cubes one obtains a simple cubic lattice. Therefore, I argue that the simple cubic lattice is the proper lattice for the discrete BKP equation. Note that a cube and simple cubic lattice are obtained as a weight diagram of a fundamental representation and a root lattice for the B_3-type Lie group, respectively.

On the other hand each of the equations in the LBTE connects f and g on a square. It does not change its form under the exchange of f and g.

As in the case of the discrete KP equation, the discrete BKP equation appears as the consistency condition for the LBTE, by arranging the LBTE in a symmetric form. In a simple cubic lattice, the fundamental regular polyhedron is a cube. Thus such symmetric equations may connect f and g on a cube. Indeed, by collecting equations in the LBTE corresponding to six squares of a cube, one can make such equations. These equations do not take the matrix times vector form. Thus the determinant-type consistency condition does not arise in this case. However, the compatibility condition may arise because one cube contains two parallel squares. Indeed, there are three ways to obtain g_{123} from g, g_1, g_2, g_3 by using the following equations:

$$
\begin{array}{lll}
1: & g, g_1, g_2 \to g_{12} & \\
 & g, g_2, g_3 \to g_{23} & g_2, g_{12}, g_{23} \to g_{123} \\
2: & g, g_2, g_3 \to g_{23} & \\
 & g, g_1, g_3 \to g_{13} & g_3, g_{23}, g_{13} \to g_{123} \\
3: & g, g_1, g_2 \to g_{12} & \\
 & g, g_1, g_3 \to g_{13} & g_1, g_{12}, g_{13} \to g_{123}.
\end{array}
\tag{16}
$$

After some calculation these procedures lead to the following expressions of g_{123}:

$$
\begin{aligned}
1: \quad g_{123} &= g \times 0 + g_1 \left(\frac{a_{31}a_{12}f_{23}}{-b_{31}b_{12}f} \right) + g_2 \left(\frac{a_{31}a_{12}f_{23}f_1}{b_{31}b_{12}f_2 f} - \frac{a_{13}a_{23}f_{12}f_3}{b_{13}b_{23}f_2 f} + \frac{f_{123}}{f_2} \right) \\
&\quad - g_3 \left(\frac{a_{13}a_{23}f_{12}}{b_{31}b_{23}f} \right) \\
2: \quad g_{123} &= g \times 0 - g_1 \left(\frac{a_{31}a_{12}f_{23}}{b_{31}b_{12}f} \right) - g_2 \left(\frac{a_{12}a_{23}f_{13}}{b_{12}b_{23}f} \right) \\
&\quad + g_3 \left(\frac{a_{12}a_{23}f_{13}f_2}{b_{12}a_{23}f_3 f} + \frac{a_{12}a_{31}f_{23}f_1}{b_{12}b_{31}f_3 f} + \frac{f_{123}}{f_3} \right).
\end{aligned}
\tag{17}
$$

These three expressions need to coincide. The condition becomes just the discrete BKP equation for f, after equating the coefficients of g, g_1, g_2, g_3.

In this case, each equation in the LBTE does not change it's form under the exchange of f and g. This means that g also needs to the satisfy the discrete BKP equation, to be able to solve the LBTE for f. Therefore, the LBTE can be solved for g only when f satisfies the discrete BKP equation, and the solution also satisfies the discrete BKP equation automatically.

To verify the existence of the solution of g to the LBTE, I construct a surface on which the initial value of g is placed and move the surface, as before. From the structure of the LBTE, such surface turns out to be constructed from the two planes $k_1 + k_2 + k_3 = c$ and $k_1 + k_2 + k_3 = c + 1$. Here c is an arbitrary integer. By using three equations in the LBTE independently, one can obtain the value of g on $k_1 + k_2 + k_3 = c + 2$. Thus, conditions to make these procedures equivalent need to be imposed on the initial value of g on the two planes $k_1 + k_2 + k_3 = c$ and $k_1 + k_2 + k_3 = c + 1$:

$$
\begin{aligned}
\frac{f_{123}}{f_3} g_3 + \frac{a_{12}}{b_{12}f_3}(b_{12}f_{12}g_{13} - f_{13}g_{12}) &= \frac{f_{123}}{f_1} g_1 + \frac{a_{23}}{b_{23}f_1}(b_{23}f_{13}g_{12} - f_{12}g_{13}) \\
&= \frac{f_{123}}{f_2} g_2 + \frac{a_{31}}{b_{31}f_2}(b_{23}f_{12}g_{23} - f_{23}g_{12}).
\end{aligned}
\tag{18}
$$

These conditions require no consistency condition. Namely one can find the initial values that satisfy these conditions. Such initial values of g provides g on $k_1 + k_2 + k_3 = c + 2$. Thus, by showing g on $k_1 + k_2 + k_3 = c + 1$ and $k_1 + k_2 + k_3 = c + 2$ automatically satisfies

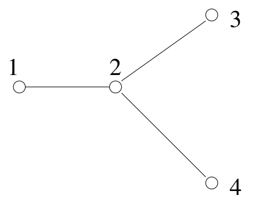

Figure 2. The Dynkin diagram for the D_4-type Lie group.

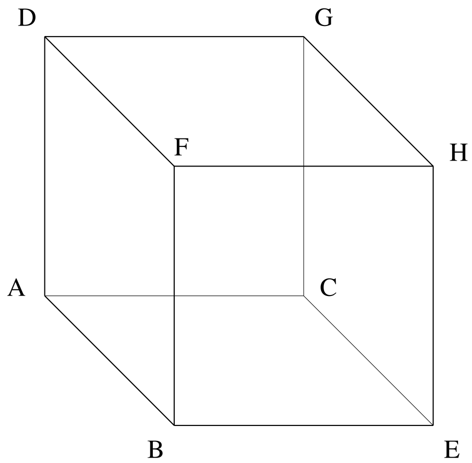

Figure 3. Compatibility condition for discrete BKP equation.

condition (18), one can verify the existence of the solution of g to the LBTE recursively. That condition becomes just the compatibility condition. As a consequence, one can verify that there is at least one solution of g for the LBTE whenever f satisfies the discrete BKP equation.

4. The DKP hierarchy case

The lowest-dimensional discrete formulae which belong to the DKP hierarchy [3] are the following two equations:

$$\begin{aligned} &z_{14}z_{23}f_{23}f_{14} - z_{13}z_{24}f_{13}f_{24} + z_{12}z_{34}f_{12}f_{34} - z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}f_{1234}f = 0 \\ &z_{23}z_{24}z_{34}f_{234}f_1 - z_{13}z_{14}z_{24}f_{134}f_2 + z_{12}z_{14}z_{24}f_{124}f_3 - z_{12}z_{13}z_{23}f_{123}f_4 = 0 \end{aligned} \tag{19}$$

where $z_{ijk} = z_{ij}z_{jk}z_{ki}$. In these equations, f represents the τ-function for the DKP hierarchy, but some variable transformation is performed (see the appendix). We call these equations discrete DKP equations.

I first seek the regular polyhedrons and proper lattice for these equations. In this case, one cannot produce such figures, since the equations are defined on a four-dimensional lattice space. However, consideration in the previous cases suggests that one can find it by considering representations of the D_4-type Lie group. Namely, the regular polyhedrons may be obtained as the weight diagrams of fundamental representations for the D_4-type Lie group, and the proper lattice may be the root lattice for the group. The Dynkin diagram for the D_4-type Lie group is shown in figure 2.

The root vectors corresponding to the indices in the figure can be represented in Cartesian coordinates as follows:

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (1, -1, 0, 0) & \boldsymbol{\alpha}_2 &= (0, 1, -1, 0) \\ \boldsymbol{\alpha}_3 &= (0, 0, 1. - 1) & \boldsymbol{\alpha}_4 &= (0, 0, 1, 1). \end{aligned} \tag{20}$$

In this choice of coordinate, weight diagrams of three fundamental representations corresponding to Dynkin indices $[1, 0, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$ become the following three four-dimensional regular 16-hedrons:

$$\begin{array}{rrr} [1, 0, 0, 0] & [0, 0, 1, 0] & [0, 0, 0, 1] \\ (1, 0, 0, 0) & (1, 1, 1, 0) & (1, 1, 1, 1) \\ (0, 1, 0, 0) & (1, 1, 0, 1) & (1, 1, 0, 0) \\ (0, 0, 1, 0) & (1, 0, 1, 1) & (1, 0, 1, 0) \\ (0, 0, 0, 1) & (0, 1, 1, 1) & (1, 0, 0, 1) \\ (-1, 0, 0, 0) & (1, 0, 0, 0) & (0, 1, 1, 0) \\ (0, -1, 0, 0) & (0, 1, 0, 0) & (0, 1, 0, 1) \\ (0, 0, -1, 0) & (0, 0, 1, 0) & (0, 0, 1, 1) \\ (0, 0, 0, -1) & (0, 0, 0, 1) & (0, 0, 0, 0). \end{array} \tag{21}$$

Two discrete DKP equations connect f on $[0, 0, 1, 0]$-type and $[0, 0, 0, 1]$-type regular 16-hedrons, respectively. Therefore, the proper lattice for the discrete DKP equations turns out to be the root lattice for the D_4-type Lie group.

Collecting the discrete formulae that depend on variables arising in the discrete DKP equations, in the modified DKP hierarchy [3], one can obtain the following eight linear equations:

$$\begin{aligned} &\sum_{jkl}\left(\tfrac{1}{2}\epsilon_{ijkl} z_{kl} f_{kl} g_j + \tfrac{1}{6} z_{jkl} f g_{jkl}\right) = 0 \\ &\sum_{jkl}\left(\tfrac{1}{2}\epsilon_{ijkl} z_{jk} f_l g_{jk} + \tfrac{1}{6} z_{jkl} f_{jkl} g\right) = 0. \end{aligned} \tag{22}$$

In each of the equations in (22), f and g are defined on two tetrahedrons dual to each other. Eight tetrahedrons arise in both equations of (22). We study how these tetraherons appear in the root lattice for the D_4-type Lie group, in order to arrange these equations in symmetric form. There are 24 tetrahedrons in the root lattice for the D_4-type Lie group. Three regular 16-hedrons in (21) contain 16 tetrahedrons. The eight tetrahedrons arising in (22) are the tetrahedrons which are simultaneously contained in the $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$ tetrahedrons. Therefore, by arranging the equations into the form in which g is on the $[0, 0, 1, 0]$ or $[0, 0, 0, 1]$ regular 16-hedrons, the symmetric arrangements may be obtained. In fact, in each case, f turns out to be on the $[0, 0, 0, 1]$ and $[0, 0, 1, 0]$ regular 16-hedrons, respectively. Moreover, the equations can be summarized in the regular matrix times vector form:

$$\begin{bmatrix} 0 & z_{34}f_{34} & z_{42}f_{24} & z_{23}f_{23} & z_{234}f & 0 & 0 & 0 \\ z_{34}f_{34} & 0 & z_{41}f_{14} & z_{13}f_{13} & 0 & z_{341}f & 0 & 0 \\ z_{24}f_{24} & z_{41}f_{14} & 0 & z_{12}f_{12} & 0 & 0 & z_{412}f & 0 \\ z_{23}f_{23} & z_{31}f_{13} & z_{12}f_{12} & 0 & 0 & 0 & 0 & z_{123}f \\ z_{234}f_{1234} & 0 & 0 & 0 & 0 & z_{34}f_{34} & z_{42}f_{24} & z_{23}f_{23} \\ 0 & z_{341}f_{1234} & 0 & 0 & z_{34}f_{34} & 0 & z_{41}f_{14} & z_{13}f_{13} \\ 0 & 0 & z_{412}f_{1234} & 0 & z_{24}f_{24} & z_{41}f_{14} & 0 & z_{12}f_{12} \\ 0 & 0 & 0 & z_{123}f_{1234} & z_{23}f_{23} & z_{31}f_{13} & z_{12}f_{12} & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_{234} \\ g_{134} \\ g_{124} \\ g_{123} \end{bmatrix} = 0 \tag{23}$$

$$\begin{bmatrix} 0 & z_{13}f_4 & z_{41}f_3 & z_{341}f_{134} & z_{34}f_1 & 0 & 0 & 0 \\ z_{12}f_4 & 0 & z_{41}f_2 & z_{412}f_{124} & 0 & z_{24}f_1 & 0 & 0 \\ z_{12}f_3 & z_{31}f_2 & 0 & z_{123}f_{123} & 0 & 0 & z_{23}f_1 & 0 \\ z_{34}f_{134} & z_{42}f_{124} & z_{23}f_{123} & 0 & 0 & 0 & 0 & z_{234}f_1 \\ z_{34}f_{234} & 0 & 0 & 0 & 0 & z_{13}f_{123} & z_{41}f_{124} & z_{341}f_2 \\ 0 & z_{24}f_{234} & 0 & 0 & z_{12}f_{123} & 0 & z_{41}f_{134} & z_{412}f_3 \\ 0 & 0 & z_{23}f_{234} & 0 & z_{12}f_{124} & z_{31}f_{134} & 0 & z_{123}f_4 \\ 0 & 0 & 0 & z_{234}f_{234} & z_{34}f_2 & z_{42}f_3 & z_{23}f_4 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} g_{12} \\ g_{13} \\ g_{14} \\ g \\ g_{34} \\ g_{24} \\ g_{23} \\ g_{1234} \end{bmatrix} = 0. \tag{24}$$

These arrangements make the lattice symmetry of the equations manifest.

In this case, the symmetric arrangements induce two determinant-type consistency conditions, since they take the matrix times vector form. After some calculations one finds that these two determinants are just the fourth power of two discrete DKP equations:

$$\begin{aligned} &(z_{14}z_{23}f_{23}f_{14} - z_{13}z_{24}f_{13}f_{24} + z_{12}z_{34}f_{12}f_{34} - z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}f_{1234}f)^4 = 0 \\ &(z_{23}z_{24}z_{34}f_{234}f_1 - z_{13}z_{14}z_{24}f_{134}f_2 + z_{12}z_{14}z_{24}f_{124}f_3 - z_{12}z_{13}z_{23}f_{123}f_4)^4 = 0. \end{aligned} \tag{25}$$

Thus, two discrete DKP equations arise as consistency conditions for the LBTE. One concludes that when one solves the equations for g, provided f is a solution to the discrete DKP equation, g also satisfies the discrete DKP equations, since the equations are invariant under the exchange of f and g.

One can prove the existence of the solution for g to the LBTE with f being a solution to the discrete DKP equations, by using the same method as in the previous cases. Namely, one can consistently evaluate g from the initial value of g defined on some surface, whenever f satisfies discrete DKP equations. First, I gather equations which remain independent after imposing the determinant-type consistency conditions. To do this, I concentrate on equations (23). When the determinant-type consistency condition for these equations is imposed, the rank of the coefficients matrix becomes 4. Thus, out of eight equations in (22), four equations remain independent. I take four such equations as follows:

$$\begin{aligned} 0:&\quad -z_{34}f_{134}g_{12} - z_{24}f_{124}g_{13} - z_{23}f_{123}g_{14} + z_{24}z_{34}z_{23}f_1g_{1234} = 0 \\ 1:&\quad z_{23}f_{234}g_{14} - z_{13}f_{134}g_{24} + z_{12}f_{124}g_{34} - z_{12}z_{23}z_{13}f_4g_{1234} = 0 \\ 2:&\quad -z_{24}f_{234}g_{13} + z_{14}f_{134}g_{23} - z_{12}f_{123}g_{34} + z_{14}z_{24}z_{12}f_3g_{1234} = 0 \\ 3:&\quad z_{34}f_{234}g_{12} - z_{14}f_{124}g_{23} + z_{13}f_{123}g_{24} - z_{14}z_{34}z_{13}f_2g_{1234} = 0. \end{aligned} \tag{26}$$

However, these equations become dependent in (24) when the determinant-type consistency condition is imposed. Namely, one equation out of four becomes a summation of the other three equations when f satisfies the discrete DKP equations. I choose the latter three equations in (26) as three such equations.

Each of the three equations allow one to evaluate, from g on $k_1 + k_2 + k_3 + k_4 = c$, g on $k_1 + k_2 + k_3 + k_4 = c + 1$. However, the conditions for g obtained by each of the LBTE to coincide, need to be imposed on the initial value of g. The conditions become

$$\begin{aligned}\frac{z_{23} f_{234} g_{14} - z_{13} f_{134} g_{24} + z_{12} f_{124} g_{34}}{-z_{12} z_{23} z_{14} f_4} &= \frac{-z_{24} f_{234} g_{13} + z_{14} f_{134} g_{23} - z_{12} f_{123} g_{34}}{z_{14} z_{24} z_{12} f_3} \\ &= \frac{z_{34} f_{234} g_{12} - z_{14} f_{124} g_{23} + z_{13} f_{123} g_{24}}{-z_{14} z_{34} z_{13} f_2}.\end{aligned} \tag{27}$$

Each equality represents the coincidence of g_{1234} obtained by the three equations in (26). These conditions require no more consistency conditions. Hence one can recursively construct the solution of g to the LBTE, when the obtained g also satisfies the same condition. This is the case, when f satisfies the discrete DKP equations. I show this for the first equality in equation (27), as an example. It can be shown by comparing the two following procedures:

$$\begin{aligned}&g(B), g(A), g(C) \to g_{12} && \\ &g(A), g(D), g(E) \to g_{23} && g_{12}, g_{23}, g_{34} \to g_{1234} \\ &g(E), g(G), g(H) \to g_{34} && \\ &g(B), g(A), g(F) \to g_{14} && \\ &g(A), g(D), g(G) \to g_{24} && g_{14}, g_{24}, g_{34} \to g_{1234} \\ &g(C), g(E), g(H) \to g_{34}. &&\end{aligned} \tag{28}$$

where,

$$\begin{aligned}&A = (0, 0, 0, 0) && B = (1, -1, 0, 0) && C = (0, -1, 1, 0) \\ &D = (-1, 1, 0, 0) && E = (-1, 0, 1, 0) && F = (0, -1, 0, 1) \\ &G = (-1, 0, 0, 1) && H = (-1, -1, 1, 1). &&\end{aligned} \tag{29}$$

Here, I substitute 1 into c for simplicity, without lose of generality. In these procedures, the value of $g(0, 0, 1, 1)$ can be obtained by using three equations in (26) independently. This requires conditions, in the form of (27), to be imposed on g on the plane $k_1 + k_2 + k_3 + k_4 = -1$. By using these conditions, one can omit $g(F)$ and $g(C)$ in (28). Equating the coefficients of the remaining five g on the plane $k_1 + k_2 + k_3 + k_4 = -1$, one can convert the condition into

$$\begin{aligned}g(G):\quad &\frac{f(0,1,1,1) f(1,0,0,0)}{z_{14}^2 z_{12} z_{13} z_{23} z_{14} z_{24} z_{34} f(-1,0,1,1) f(0,0,0,1)} \\ &\times \left[(1 - \mathrm{e}^{-\partial_1 - \partial_2}) \left(\frac{z_{14} z_{34} z_{13} f_2 f_{134} - z_{14} z_{24} z_{12} f_3 f_{124} + z_{pq} z_{13} z_{23} f_{123} f_4}{f_{234} f_1} \right) \right] \\ &= 0\end{aligned} \tag{30}$$

$$\begin{aligned}g(E):\quad &\frac{f(0,1,1,1) f(1,0,0,0)}{z_{14} z_{24} z_{34} z_{12} z_{13}^2 z_{23} f(-1,0,0,0) f(0,0,1,0)} \\ &\times \left[(1 - \mathrm{e}^{-\partial_1 - \partial_2}) \left(\frac{z_{13} z_{14} z_{34} f_2 f_{134} + z_{12} z_{14} z_{24} f_{124} f_3 - z_{12} z_{13} z_{23} f_4 f_{123}}{f_1 f_{234}} \right) \right] \\ &= 0\end{aligned} \tag{31}$$

$$g(A):\quad \frac{f(0,1,1,1) f(1,0,1,1)}{z_{14} z_{24} z_{12}^2 z_{23} z_{13} f(0,0,0,1) f(0,0,1,0)}$$

$$
\begin{aligned}
&\times\left[(\mathrm{e}^{-\partial_1}-\mathrm{e}^{\partial_2})\left(\frac{z_{14}z_{23}f_{14}f_{23}-z_{24}z_{13}f_{13}f_{24}+z_{34}z_{12}f_{12}f_{34}}{ff_{1234}}\right)\right]\\
&+\frac{f(0,1,1,1)f(1,0,0,0)f(-1,0,1,1)}{z_{14}^2z_{24}z_{12}z_{13}^2z_{23}f(-1,0,0,0)f(0,0,0,1)f(0,0,1,0)}\\
&\times\left[(1-\mathrm{e}^{-\partial_1-\partial_2})\left(\frac{-z_{14}z_{24}z_{12}f_{124}f_3+z_{12}z_{23}z_{13}f_{123}f_4+z_{14}z_{34}z_{13}f_{134}f_2}{f_1f_{234}}\right)\right]\\
=\ &0. \qquad (32)
\end{aligned}
$$

All these equations are satisfied if f satisfies two discrete DKP equations. The equality of equations (2) and (3) or (1) and (3) in (27), can be shown by using a similar approach. Thus, it is shown that, whenever f satisfies the discrete DKP equations, there is at least one solution of g to the LBTE.

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Appendix A. Discrete formula for the integrable hierarchy

The integral identity known as the bilinear identity [5, 6], leads all the equations satisfied by the τ-function of the integrable hierarchy. The discrete formulae that appear in this paper are also obtained from this identity. In this appendix, for completeness, I briefly explain the way to obtain the discrete formulae from the bilinear identities.

A.1. Bilinear identity for the KP hierarchy and the modified KP hierarchy

The bilinear identity for the KP hierarchy is as follows [5]:

$$\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau(x-\epsilon(z^{-1}))\tau(x'+\epsilon(z^{-1}))=0. \qquad (33)$$

Here, x and x' are infinite-dimensional vectors embodying an infinite number of variables in the KP hierarchy:

$$x=(x_1,x_2,x_3,\ldots). \qquad (34)$$

These variables are continuous variables in the sense that they become variables for differential equations contained in the KP hierarchy. $\epsilon(z)$ is a vector represented as

$$\epsilon(z)=(z,\tfrac{1}{2}z^2,\ldots) \qquad (35)$$

ξ is a function of x and z represented as

$$\xi(x,z)=\sum_{n=1}^{\infty}\frac{1}{n}x_nz^n. \qquad (36)$$

By choosing $x-x'$ properly, one can obtain all the equations satisfied by the τ-function for the KP hierarchy.

An infinite number of discrete variables are defined through the Miwa transformation [5] from continuous variables (34):

$$\frac{\partial}{\partial k_i}=\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial x_n}z_i^n. \qquad (37)$$

Here, z_i are arbitrary complex constants, and discrete variables exist for each different z_i.

The difference equations contained in the KP hierarchy depend on these variables.

The difference equation itself is obtained by expanding the bilinear identity, after substituting the appropriate summation of $\epsilon(z_1), \epsilon(z_2), \ldots$ into $x - x'$. The dimension of the equations increases with the number of terms contained in the summation. The most simple nontrivial equation within it is obtained when one substitutes $\epsilon(z_1) + \epsilon(z_2) + \epsilon(z_3)$ into $x - x'$. In this case, after some calculations one obtains the following equation:

$$z_1 z_{23} \tau_{k_1} \tau_{k_2 k_3} + z_2 z_{31} \tau_{k_2} \tau_{k_3 k_1} + z_3 z_{12} \tau_{k_3} \tau_{k_1 k_2} = 0. \tag{38}$$

Hence, the lowest-dimensional discrete equation contained in the KP hierarchy is a three-dimensional equation. Another three-dimensional equation is obtained, when one substitutes $\epsilon(z_1) + \epsilon(z_2) + \epsilon(z_3) - \epsilon(z_4)$ into $x - x'$. This is the discrete KP equation employed in this paper. Replacing z_{i4} by $z_i\,(i = 1, 2, 3)$, after suitable variable transformation, one can convince oneself that the discrete KP equation is the same as equation (38).

The bilinear identity for the modified KP hierarchy can be obtained as a modification of the KP hierarchy's one [5]. It takes the following form:

$$\mathrm{Res}_{z=0}\,\mathrm{d}z\, z \mathrm{e}^{\xi(x,z)-\xi(x',z)} \tau'(x - \epsilon(z)) \tau(x' + \epsilon(z)) = 0. \tag{39}$$

By substituting arbitrary sums of three out of $\epsilon(z_1), \epsilon(z_2), \epsilon(z_3), \epsilon(z_4)$ into $x - x'$ this identity leads to four equations the same as those in the LBTE for the discrete KP equation.

A.2. Bilinear identity for the BKP hierarchy and the modified BKP hierarchy

The τ-function for the BKP hierarchy satisfies the following bilinear identity [5]:

$$\mathrm{Res}_{z=0}\,\mathrm{d}z \frac{1}{z} \mathrm{e}^{\tilde{\xi}(x,z)-\tilde{\xi}(x',z)} \tau(x - 2\tilde{\epsilon}(z^{-1})) \tau(x + 2\tilde{\epsilon}(z^{-1})) = \tau(x)\tau(x'). \tag{40}$$

Here, x, x', $\tilde{\epsilon}$ and $\tilde{\xi}$ are the same as those for the KP hierarchy except for the absence of variables of even number indices. i.e.

$$\begin{aligned} &x = (x_1, x_3, x_5, \ldots) \qquad \tilde{\epsilon}(z) = (z, \tfrac{1}{3}z^3, \ldots) \\ &\tilde{\xi}(x, z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} x_{2n-1} z^{2n-1}. \end{aligned} \tag{41}$$

By expanding this bilinear formula, one can obtain all the equations contained in the BKP hierarchy.

The dependent variables for discrete equations which belong to the BKP hierarchy are difined through the following variable transformation:

$$\frac{\partial}{\partial k_i} = 2 \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{\partial}{\partial x_{2n-1}} z_i^{2n-1}. \tag{42}$$

The discrete equations themselves are obtained by substituting the summation of $\tilde{\epsilon}(z_1), \tilde{\epsilon}(z_2), \tilde{\epsilon}(z_3) \ldots$ into $x - x'$. The most simple nontrivial equation is obtained when one substitutes $\tilde{\epsilon}(z_1) + \tilde{\epsilon}(z_2) + \tilde{\epsilon}(z_3)$ into $x - x'$. The equation is the discrete BKP equation.

The bilinear identities for the modified BKP hierarchy can be obtained as a modification of those for the BKP hierarchy and take the following form [6]:

$$\mathrm{Res}_{z=0}\,\mathrm{d}z \frac{1}{z} \mathrm{e}^{\tilde{\xi}(x,z)-\tilde{\xi}(x',z)} \tau'(x - 2\tilde{\epsilon}(z^{-1})) \tau(x + 2\tilde{\epsilon}(z^{-1})) = 2\tau(x)\tau'(x') - \tau'(x)\tau(x). \tag{43}$$

By substituting the summation of two of $\tilde{\epsilon}(z_1), \tilde{\epsilon}(z_2), \tilde{\epsilon}(z_3)$ into $x - x'$, one can obtain the LBTE for the discrete BKP equation used in this paper.

A.3. Bilinear identity for the fermionic BKP hierarchy

The bilinear identity for the fermionic BKP hierarchy is [3]

$$\begin{aligned}&\tfrac{1}{2}((-1)^{n+m}-1)\tau_n(x)\tau_m(x') + \mathrm{Res}\,\mathrm{d}z\, z^{n-m-2}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau_{n-1}(x-\epsilon(z))\tau_{m+1}(x'+\epsilon(z))\\ &\qquad + \mathrm{Res}\,\mathrm{d}z\, z^{m-n-2}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau_{n+1}(x+\epsilon(z))\tau_{m-1}(x'-\epsilon(z)) = 0. \qquad (44)\end{aligned}$$

where x, x', $\epsilon(z)$ and $\xi(x, z)$ are same as those of the KP hierarchy, and n represents the charge in the free fermion description [5, 6]. Variables for the discrete equation are also the same as those of the KP hierarchy. All the equations which belong to the BKP hierarchy can be obtained by appropriately choosing $x - x'$ and $n - m$. The most simple nontrivial discrete equation is obtained when $x - x' = \epsilon(z_1) + \epsilon(z_2)$ and $n - m = 1$ or 3, and is

$$\begin{aligned}&z_2\tau_{n+1}(p+1,q)\tau_{n+2}(p,q+1) + z_1\tau_{n+1}(p,q+1)\tau_{n+2}(p+1,q)\\ &\qquad + z_{12}\tau_{n+1}(p,q)\tau_{n+2}(p+1,q+1)\\ &\qquad + z_{12}z_1z_2\tau_n(p,q)\tau_{n+3}(p+1,q+1) = 0. \qquad (45)\end{aligned}$$

This equation is a three-dimensional equation when one considers n as one variable. To place the three variables on an equal footing, I transform them as follows:

$$p = k_1 \qquad q = k_2 \qquad n = p + q. \qquad (46)$$

In these variables, the equation is expressed as

$$z_2 f_1 f_{23} + z_1 f_2 f_{13} + z_{12} f_3 f_{12} + z_{12}z_2z_1 f f_{123} = 0. \qquad (47)$$

This equation a is three-dimensional discrete equation. However, another three-dimensional equation can be obtained when $x - x' = \epsilon(z_1) + \epsilon(z_2) - \epsilon(z_3)$ and $n - m = 1$. The equation obtained is the discrete fermionic BKP equation employed in this paper.

The bilinear identity for the modified fermionic BKP hierarchy is

$$\begin{aligned}&\mathrm{Res}\,\mathrm{d}z\, z^{n-m-2}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau'_{n-1}(x-\epsilon(z))\tau_{m+1}(x'+\epsilon(z))\\ &\qquad + \mathrm{Res}\,\mathrm{d}z\, z^{m-n-2}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau'_{n+1}(x+\epsilon(z))\tau_{m-1}(x'-\epsilon(z))\\ &\quad = -\tfrac{1}{2}((-1)^{n+m}-1)\tau'_n(x)\tau_m(x') + \tau_n(x)\tau'_m(x'). \qquad (48)\end{aligned}$$

By substituting arbitrary sums of two of $\epsilon(z_1)$, $\epsilon(z_2)$, $\epsilon(z_3)$ into $x - x'$, one can obtain three equations in the LBTE for the discrete fermionic BKP equation.

A.4. Bilinear identity for the DKP hierarchy

The bilinear identity for the DKP hierarchy is as follows [3]:

$$\begin{aligned}&\mathrm{Res}_{z=0}\,\mathrm{d}z\, z^{n-m-2}\mathrm{e}^{\xi(x,z)-\xi(x',z)}\tau_{n-1}(x-\epsilon(z^{-1}))\tau_{m+1}(x'+\epsilon(z^{-1}))\\ &\qquad + \mathrm{Res}_{z=0}\,\mathrm{d}z\, z^{m-n-2}\mathrm{e}^{-\xi(x,z)+\xi(x',z)}\tau_{n+1}(x+\epsilon(z^{-1}))\tau_{m-1}(x'-\epsilon(z^{-1})) = 0. \quad (49)\end{aligned}$$

Here, n represents the charge in the free fermion description. x, x', ϵ and ξ are the same as the KP hierarchy. The dependent variables for the discrete equations which belong to the DKP hierarchy are also the same as the KP hierarchy. All the equations which belong to the DKP hierarchy can be obtained by expanding this identity, when one chooses $n - m$ and $x - x'$ appropriately. The lowest-dimensional discrete equations within it are obtained when $x - x' = \epsilon(z_1) + \epsilon(z_2) + \epsilon(z_3)$ and $n - m = 2, 4$ or $x - x' = \epsilon(z_1) + \epsilon(z_2) - \epsilon(z_3)$ and $n - m = 0, 2$. However, two of these equations overlap with other two equations. As a consequence, the following two equations turn out to be the lowest-dimensional discrete DKP equations:

$$
\begin{aligned}
&z_p z_{qr} f_n(p, q+1, r+1) f_n(p+1, q, r) + z_q z_{rp} f_n(p+1, q, r+1) f_n(p, q+1, r)\\
&\qquad + z_r z_{pq} f_n(p+1, q+1, r) f_n(p, q, r+1)\\
&\qquad - z_p z_q z_r z_{pq} z_{qr} z_{rp} f_{n+2}(p+1, q+1, r+1) f_{n-2}(p, q, r) = 0\\
&z_q z_r z_{qr} f_{n-1}(p, q+1, r+1) f_{n-3}(p+1, q, r) + z_r z_p z_{rp} f_{n-1}(p+1, q, r+1) f_{n-3}(p, q+1, r)\\
&\qquad + z_p z_q z_{pq} f_{n-1}(p+1, q+1, r) f_{n-3}(p, q, r+1)\\
&\qquad + z_{pq} z_{qr} z_{rp} f_{n-1}(p+1, q+1, r+1) f_{n-3}(p, q, r) = 0.
\end{aligned}
\tag{50}
$$

These equations are four-dimensional equations, considering n as one variable. To place the four variables on an equal footing, I transform them as follows:

$$
p = k_1 \qquad q = k_2 \qquad r = k_3 \qquad n = k_1 + k_2 + k_3 + k_4. \tag{51}
$$

In these four variables, the equations take the following forms:

$$
\begin{aligned}
&z_1 z_{23} f_{23} f_{14} - z_{13} z_2 f_{13} f_{24} + z_{12} z_3 f_{12} f_{34} - z_{12} z_{13} z_1 z_{23} z_2 z_3 f_{1234} f = 0\\
&z_{23} z_2 z_3 f_{23} f_1 - z_{13} z_1 z_2 f_{134} f_2 + z_{12} z_1 z_2 f_{124} f_3 - z_{12} z_{13} z_{23} f_{123} f_4 = 0.
\end{aligned}
\tag{52}
$$

However, one can obtain other four-dimensional equations for the DKP hierarchy as in the previous case. One of which is obtained when $x - x' = \epsilon(z_1) + \epsilon(z_2) + \epsilon(z_3) - \epsilon(z_4)$ and $n - m = 2$, and is the first equation in the discrete DKP equations employed in this paper. Another one is obtained when $x - x' = \epsilon(z_1) + \epsilon(z_2) + \epsilon(z_3) + \epsilon(z_4)$ and $n - m = 4$, and is the second equation in the discrete DKP equation.

Modification of the bilinear identity for the discrete DKP equation leads to [3]

$$
\begin{aligned}
&\mathrm{Res}_{z=0}\, \mathrm{d}z\, z^{n-m-2} \mathrm{e}^{\xi(x,z)-\xi(x',z)} \tau'_{n-1}(x - \epsilon(z^{-1})) \tau_{m+1}(x' + \epsilon(z^{-1}))\\
&\qquad + \mathrm{Res}_{z=0}\, \mathrm{d}z\, z^{m-n-2} \mathrm{e}^{-\xi(x,z)+\xi(x',z)} \tau'_{n+1}(x + \epsilon(z^{-1})) \tau_{m-1}(x' - \epsilon(z^{-1}))\\
&\quad = \tau_n(x) \tau'_m(x').
\end{aligned}
\tag{53}
$$

This identity is the bilinear identity for the modified DKP equation. By taking arbitrary summations of three of $\epsilon(z_1)$, $\epsilon(z_2)$, $\epsilon(z_3)$, $\epsilon(z_4)$, one can obtain eight equations in the LBTE for the discrete DKP equations.

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